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Yang–Mills effective action at high temperature

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Abstract. Yang–Mills theory undergoes a transition from a confined to a deconfined phase in the intermediate temperature regime, where perturbation is not applicable. In order to approach this phase transition from the high temperature side we study the effective action for the eigenvalues of the order parameter, the Polyakov loop. By means of a covariant derivative expansion we integrate out fast varying quantum fluctuations around background gluon fields and assume that these are slowly varying, but that the amplitude of A_4 is arbitrary. Our results can be used to study correlation functions of the order parameter at high temperatures.

1 Introduction

The equilibrium behavior of quantum field theories at finite temperature is described through the grand canonical partition function. This introduces a compactification of the imaginary time direction in the Euclidean formulation of the theory. The partition function of a statistical system is defined as a sum over physical states n:

$$Z = \sum_{n_{\rm phys}} \langle n | e^{-\beta \mathcal{H}} | n \rangle, \tag{1}$$

where $\beta = 1/T$ and $\mathcal{H}$ is the Hamiltonian of the system. In Yang–Mills theory physical states are those which are gauge-invariant, i.e. invariant under gauge transformation of the gluon fields:

$$A_\mu(x) \to [A_\mu(x)]^{\Omega(x)} = \Omega(x)^\dagger A_\mu(x)\, \Omega(x) + i\Omega(x)^\dagger \partial_\mu \Omega(x), \quad \Omega(x) = \exp\{i\,\omega_a(x) t^a\}\,. \tag{2}$$

The partition function in its Euclidean-invariant form is given by

$$Z = \int DA_\mu \exp\left\{ -\frac{1}{4g^2} \int_0^{\beta=\frac{1}{T}} dt \int d^3x\, F^a_{\mu\nu} F^a_{\mu\nu} \right\}. \tag{3}$$

Due to the compactified time direction the gluon fields obey periodic boundary conditions in time

$$A_\mu(0,x) = A_\mu(\beta,x). \tag{4}$$

There are special gauge transformations, which leave the periodic boundary conditions for the gluons intact, but which themselves are only periodic up to an element of the center of the gauge group. The center group $Z(N_c)$ is a discrete one and has the elements

$$z_k = e^{2\pi i k/N_c} \qquad \text{where} \qquad k \in \{0, N_c - 1\}. \tag{5}$$

In particular for the gauge group $SU(2)$, which we will use for our calculations, the elements of $Z(2)$ are

$$z_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad z_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

The center group is of relevance for a description of the order parameter of the confinement-deconfinement phase transition. It is given by $< {\rm Tr} P >$ where P is the Polyakov line

$$P(x) = \mathcal{P} \exp\left(i \int_0^{1/T} dt\, A_4 \right). \tag{7}$$

Here $\mathcal{P}$ stands for path ordering. The Polyakov line is not invariant under $Z(N_c)$ transformations but transforms as

$$P(x) \to z_k^{-1} P(x). \tag{8}$$

One sees immediately that a manifest $Z(N_c)$ symmetry implies $< {\rm Tr} P >= 0$. This corresponds to the confined phase. If $< {\rm Tr} P > \neq 0$ then the symmetry must have been broken spontaneously. This corresponds to the deconfined phase. (In the presence of fermions the symmetry gets broken explicitly.) At very high temperatures the potential energy of the Polyakov line (or of A_4) has its zero-energy minima for values of $P(x)$ at the center of the gauge group (or for quantized values of A_4). High temperature perturbation theory hence corresponds to the system oscillating around these trivial values of the Polyakov line, *i.e.* $< {\rm Tr}\, P > \neq 0$. As the temperature decreases, however, the fluctuations of the Polyakov line increase and eventually at the critical temperature T_c the system undergoes a phase transition from a deconfined to a confined phase which has $< {\rm Tr}\, P >= 0$. In order to approach this phase transition from the high-temperature side, one needs to study the

Polyakov line in its whole range of possible variation. In our recent paper [1] we worked with static and diagonal A_4 gluon fields and the gauge group $SU(2)$. In this case the Polyakov line

$$P(x) = \exp\left(i\frac{A_4(x)}{T}\right) \tag{9}$$

has the gauge invariant eigenvalues

$$e^{\pm i\pi\nu} \qquad \text{where} \qquad \nu = \sqrt{A_4^a A_4^a}/2\pi T. \tag{10}$$

We assume that the gluons are varying slowly, but we allow for an arbitrary amplitude of the A_4 fields. We then find the non-trivial effective action for the eigenvalues of the Polyakov line, interacting in a covariant way with the spatial gluon fields A_i.

2 Energy scales at high temperatures

Since the gluon fields are periodic in time one can make a Fourier decomposition:

$$A_\mu(t,x) = \sum_{k=-\infty}^{\infty} A_\mu(\omega_k,x)\,e^{i\omega_k t}, \qquad \omega_k = 2\pi k\,T, \tag{11}$$

where the Matsubara frequencies ω_k are the quantized values for the energies. The energy scales that show up in the theory are at the tree level the temperature T, on the quantum level the Debye mass scale gT which arises from screening effects for the color-electric gluons, and g^2T, which is the (non-perturbative) scale of the color-magnetic gluons. The first step on the way to an effective theory is to integrate out the non-zero Matsubara modes at the tree level, since they become very heavy at high temperatures. This reduces the original 4D Euclidean symmetry to a 3D static one. The next step is to include quantum fluctuations. There all Matsubara modes show up in loops again and produce infinitely many effective vertices. In [1] we obtained all these vertices, but restricted to low momenta $p < T$.

3 The 1-loop action

We use a background field method where the gluon fields are decomposed into a static background field (denoted by a bar) and time-dependent quantum fluctuations around them:

$$A_\mu = \bar{A}_\mu + a_\mu. \tag{12}$$

For the quantum fluctuations we choose the background Lorenz gauge $D_\mu^{ab}(\bar{A})\,a_\mu^b = 0$, where D_μ is the covariant derivative in the adjoint representation. A one loop calculation corresponds to expanding the action to quadratic power in a_μ. This results in the following effective theory for the background $\bar{A}$ fields:

$$Z(\bar{A}) = e^{\bar{S}}\int Da D\chi D\chi^{+} \exp\left[-\frac{1}{2g^2(M)} \int d^4x a_\mu^b W_{\mu\nu}^{bc} a_\nu^c - \int d^4x \chi^{+a} D^2 \chi^a\right] . \tag{13}$$

Here χ, χ^+ are the ghost fields and

$$\bar{S} = -\frac{1}{4g^2(M)}\int d^4x F_{\mu\nu}^a(\bar{A}) F_{\mu\nu}^a(\bar{A}) \tag{14}$$

is the action of the background fields. The quadratic form for a_μ is given by

$$W_{\mu\nu}^{ab} = -[D^2(\bar{A})]^{ab}\delta_{\mu\nu} - 2f^{acb}F_{\mu\nu}^c(\bar{A})\ . \tag{15}$$

Integrating out a, χ and χ^+ provides us with the 1-loop action

$$S_{1-\text{loop}} = \log\left(\det W\right)^{-1/2} + \log\det\left(-D^2\right)\ . \tag{16}$$

Since the only gluon fields which are left are the background fields we will omit the bar from now on.

4 Gradient expansion of $S_{1-\text{loop}}$

For the background $A_4(x)$ fields one can always choose a gauge where they are static and diagonal in the fundamental representation, while the spatial A_i components are generally speaking time dependent. We shall, however, assume time-independence for all the background components. The prize for this is the loss of invariance under certain residual time-dependent gauge transformations, which we shall discuss later on.

We then expand the 1-loop action in powers of the spatial covariant derivative D_i and obtain the kinetic energy by identifying the electric and magnetic fields as

$$[D_i, D_4] = -iF_{i4} = -iE_i,$$
$$B_i = \frac{1}{2}\epsilon_{ijk}F_{jk} = \frac{i}{2}\epsilon_{ijk}[D_j, D_k]. \tag{17}$$

For $SU(2)$ there are only two independent color vectors in the electric (magnetic) sector, E_i (B_i) and A_4, and we thus expect the following structure for the kinetic energy to quadratic order in the electric and magnetic fields:

$$S_{1-\text{loop}} = \int\frac{d^3x}{T}\left[-T^4\,V(A_4^2) + E_i^2 f_1(A_4^2) + \frac{(E_iA_4)^2}{A_4^2}f_2(A_4^2) \right.$$
$$\left. B_i^2\,h_1(A_4^2) + \frac{(B_iA_4)^2}{A_4^2}h_2(A_4^2) + \ldots\right]. \tag{18}$$

The potential energy $V(A_4^2)$ has long been known [2,3], the functions $f_{1,2}, h_{1,2}$ from the kinetic energy were obtained in [1].

4.1 The proper time formalism

The functional determinants in the 1-loop action eq. (16) are UV divergent which reflects the running of the coupling constant. As a regularization we hence introduce a Pauli-Villars cutoff M. In addition we want to normalize the functional determinants with respect to the free theory. This can be done with a method introduced by

Schwinger [4], which yields for the ghost functional determinant

$$\log\det(-D^2)_{\rm Norm,Reg} \equiv \log \frac{\det(-D_\mu^2)}{\det(-\partial_\mu^2)}\,\frac{\det(-\partial_\mu^2+M^2)}{\det(-D_\mu^2+M^2)}$$
$$= -\int_0^\infty \frac{ds}{s}\,{\rm Sp}\left[\left(1-e^{-sM^2}\right)\left(e^{sD_\mu^2}-e^{s\partial_\mu^2}\right)\right]\,, \quad (19)$$

where Sp denotes a functional trace. The trace can be taken by inserting a plane wave basis:

$$\log\det(-D^2)_{\rm Norm,Reg} =$$
$$-\int d^3x \sum_{k=-\infty}^{\infty}\int\frac{d^3p}{2\pi^3}\int_0^\infty\frac{ds}{s}\left(1-e^{-sM^2}\right)$$
$$\times{\rm Tr}\left\{\exp\left[s(\mathcal{A}^2+(D_i+ip_i)^2)\right]\right.$$
$$\left.-\exp\left[-s(\omega_k^2+p^2)\right]\right\}, \quad (20)$$

where we defined the adjoint matrix $\mathcal{A}^{ab} = f^{acb}A_4^c + i\omega_k\delta^{ab}$. A similar result can be found for the ghost functional determinant, see [1] for details.

4.2 The effective potential

We find the potential $V(A_4^2)$ at zeroth order in D_i. This corresponds to $\mathbf{E}=\mathbf{B}=\mathbf{0}$ and

$$\det{}^{-\frac{1}{2}}W_{\mu\nu} = \det{}^{-2}(-D_\mu^2). \quad (21)$$

We choose the gauge where A_4 is diagonal in the fundamental representation:

$$A_4^a = \delta^{a3}\phi = \delta^{a3}2\pi T\,\nu \quad , \quad \nu = \frac{\sqrt{A_4^aA_4^a}}{2\pi T}. \quad (22)$$

The resulting potential is well known [2,3] and reads

$$V = \frac{1}{3(2\pi)^2T^4}\,\phi^2(2\pi T-|\phi|)^2|_{\rm mod\;2\pi T}$$
$$= \frac{(2\pi)^2}{3}\,\nu^2(1-\nu)^2|_{\rm mod\,1}. \quad (23)$$

It is shown in Fig. (1). The potential is clearly periodic in ν with period one, which means that it is center-symmetric. At the minima of the potential A_4 has quantized values. For the Polyakov line this means that it assumes values of $Z(2)$. In particular for $SU(2)$ with

$$P = \exp\left(iA_4^a\frac{\tau^a}{2T}\right) = \cos\frac{|A_4|}{2T} + i\frac{A_4^a\tau^a}{|A_4|}\sin\frac{|A_4|}{2T}, \quad (24)$$

the minima correspond to

$$\nu = 0,2,\ldots \longrightarrow P = \begin{pmatrix}1&0\\0&1\end{pmatrix}, \quad (25)$$

$$\nu = 1,3,\ldots \longrightarrow P = \begin{pmatrix}-1&0\\0&-1\end{pmatrix}. \quad (26)$$

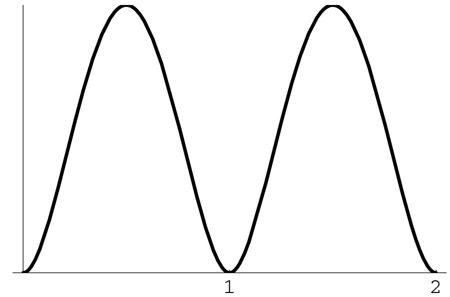

Fig. 1. The periodic potential V with period 1 in units of ν.

At the minima one hence has $<{\rm Tr\,P}>\neq 0$, and at high temperatures perturbation theory is performed around one of these center group values. At lower temperatures, however, the fluctuations of $<{\rm Tr\,P}>$ increase and eventually at the phase transition point $<{\rm Tr\,P}>\to 0$. It is hence of interest to study the fluctuations of the Polyakov line beyond the perturbative minima. This is tantamount to calculating the kinetic energy, which we shall do by means of an expansion in higher powers of the covariant derivative.

4.3 Higher powers of the covariant derivative

For the kinetic energy we encounter structures of the type

$$\exp s\big(\mathcal{A}^2+(D_i+ip_i)^2\big)\,, \quad \mathcal{A}^{ab} = f^{acb}A_4^c + i\omega_k\delta^{ab} \quad (27)$$

which we have to expand in powers of D_i. This can be done by using

$$e^{A+B} = e^A + \int_0^1 d\alpha\, e^{\alpha A}\,B\,e^{(1-\alpha)A} \quad (28)$$
$$+\int_0^1 d\alpha\int_0^{1-\alpha}d\beta\,e^{\alpha A}\,B\,e^{\beta A}\,B\,e^{(1-\alpha-\beta)A}+\ldots,$$

and dragging $B = D_i, D_i^2$ to the right with the help of

$$[B,e^A] = \int_0^1 d\gamma\,e^{\gamma A}\,[B,A]\,e^{(1-\gamma)A}. \quad (29)$$

Then we have to evaluate all the integrals over $\alpha,\beta,\gamma,\ldots$, p, s and sum over the Matsubara frequencies $\omega_k = 2\pi kT$. This should be done separately for the ghost and gluon determinants. For the electric sector one has to go to second order in D_i and for the magnetic sector to the quartic order. We will only show the results of the calculations. The details can be found in [1].

4.4 Results for the electric sector

We indeed find the structure for the kinetic energy that was outlined in eq. (18) with the functions given by

$$f_1(\nu) =$$
$$\frac{11}{48\pi^2}\left[2\left(\log\mu-\gamma_{\rm E}\right)-\psi\left(-\frac{\nu}{2}\right)-\psi\left(\frac{\nu}{2}\right)+\frac{20}{11\nu}\right],$$
$$\nu = \frac{\sqrt{A_4^aA_4^a}}{2\pi T} \quad (30)$$

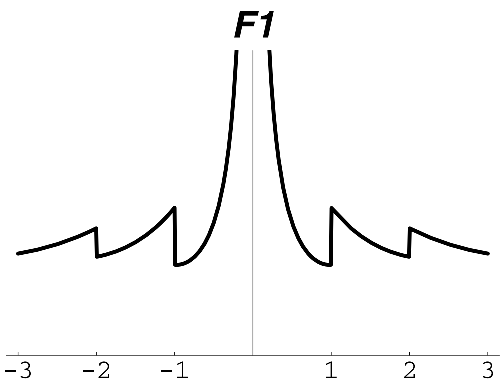

Fig. 2. The function $f_1(\nu)$ with the constant part subtracted, in different intervals.

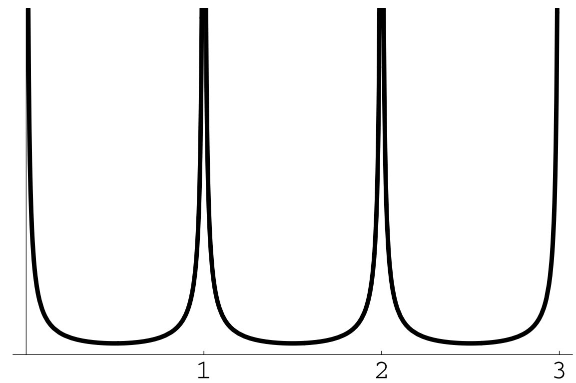

Fig. 3. The symmetric function in f_3, $h_{1,2}$ without the constant part, in different intervals.

$$f_2(\nu) = \frac{11}{48\pi^2}\left[\psi\left(-\frac{\nu}{2}\right)+\psi\left(\frac{\nu}{2}\right)-\psi(\nu)-\psi(1-\nu)-\frac{20}{11\nu}\right]. \tag{31}$$

Here ψ is the digamma function,

$$\psi(z) = \frac{\partial}{\partial z}\log\Gamma(z)\,, \tag{32}$$

γ_{E} is the Euler constant and μ is a UV cutoff that we introduced in the sum over Matsubara frequencies. It is related to the Pauli-Villars mass as

$$\mu = \frac{M}{4\pi T}\,e^{\gamma_{\mathrm{E}}}\,. \tag{33}$$

This scale has been previously found in [5] for the running coupling constant in the dimensionally reduced theory, and our result agrees. It should be noted here that the above results are valid for $0 \le \nu \le 1$. In other intervals the functional forms of f_1 and f_2 are different. We show the results for f_1 for a broader range of ν in Fig. (2). One can clearly see that this result is not $Z(2)$ symmetric, the same is true for f_2. However, one particular combination, namely

$$\begin{aligned} f_3(\nu) &\equiv f_1(\nu)+f_2(\nu) \\ &= \frac{11}{48\pi^2}\left[2\left(\log\mu-\gamma_{\mathrm{E}}\right)-\psi(\nu)-\psi(1-\nu)\right] \end{aligned} \tag{34}$$

turns out to periodic. We plot it in Fig. (3). The reason for this is the following: We chose the gauge where A_4 is static and diagonal in the fundamental representation. This leaves certain residual gauge symmetries:

$$\begin{aligned} &A_\mu \to S^\dagger A_\mu S + iS^\dagger\partial_\mu S, \\ &S(x,t) = \exp\left\{-i\frac{\tau^3}{2}\left[\alpha(x)+2\pi t T n\right]\right\}. \end{aligned} \tag{35}$$

Our invariants in the electric sector can be expressed as

$$E_i^a E_i^a f_1 + \frac{(E_i^a A_4^a)^2}{A_4^b A_4^b} f_2 = E_i^{\|} E_i^{\|} f_3 + E_i^{\perp} E_i^{\perp} f_1 \tag{36}$$

where $E_i^{\|} E_i^{\|} = (E_i^1)^2 + (E_i^2)^2$ and $E_i^{\perp} E_i^{\perp} = (E_i^3)^2$ denote the structures parallel and orthogonal to A_4^2.

The time-dependent gauge transformations eq. (35) now introduce large time derivatives in the $A_i^{1,2}$ but not in the A_i^3 fields. Since our background fields are static we do not have invariance under eq. (35). The time-dependence enters in $E_i^{\perp} E_i^{\perp}$, but not in $E_i^{\|} E_i^{\|}$. Hence one should not expect gauge-invariance in the structure $E_i^{\perp} E_i^{\perp} f_1$, since it is only quadratic in $\dot{A}_i^{1,2}$. However, even for time-dependent background fields one would have to sum over all powers $\dot{A}_i^{1,2}/T$. This is a challenging problem beyond the scope of this work.

4.5 Results for the magnetic sector

In order to obtain the functions $h_{1,2}$ from eq. (18) we have to go to quartic order in D_i. At this order there are also mixing terms between the electric and the magnetic field. We ignore these since we are only interested in the kinetic energy in the magnetic sector. Again the functional form of $h_{1,2}$ depends on the interval that we choose for A_4. For $0 \le \nu \le 1$ we find

$$h_1(\nu) = \frac{11}{96\pi^2}\left[4\left(\log\frac{M}{4\pi T}+\frac{\gamma_{\mathrm{E}}}{2}\right)-\psi(\nu)-\psi(1-\nu)\right], \tag{37}$$

$$h_2(\nu) = -\frac{11}{96\pi^2}\left[2\gamma_{\mathrm{E}}+\psi(\nu)+\psi(1-\nu)\right]. \tag{38}$$

In Fig. (3) we plot the constant part (which is the same as for f_3) for different intervals. The result is obviously center-symmetric.

4.6 Renormalization

The functions f_1 and h_1 are UV divergent, they contain the Pauli-Villars cutoff M in the subtraction scale μ, see eq. (33). This divergence is expected and necessary to cancel the tree level divergence from the running coupling constant:

$$-\frac{F^a_{\mu\nu}\,F^a_{\mu\nu}}{4g^2(M)} = -F^a_{\mu\nu}\,F^a_{\mu\nu}\,\frac{11}{3}\mathrm{N_c}\frac{1}{32\pi^2}\log\frac{M}{\Lambda}\ . \qquad (39)$$

If we evaluate the coupling constant at the scale M and add the tree level action eq. (39) to our 1-loop action, then we obtain a finite result. In the effective action the scale M in μ gets replaced by Λ.

5 Comparison to previous work

In reference [6] a covariant derivative expansion of the 1-loop Yang-Mills action is performed. While we keep all powers of the background A_4 field the author of [6] goes only to quadratic order. For a comparison we have to expand our functions $f_{1,2}$ and $h_{1,2}$ to quadratic order in ν and we find that the results agree.

We mentioned in the section on the electric sector that one combination of our functions, namely $f_3 = f_1 + f_2$ is $Z(2)$ symmetric. This function has been obtained in [7] in the context of a calculation of the interface tension of $Z(N)$ instantons, and our result again agrees.

6 Summary

In our recent paper [1] we studied the effective action for the eigenvalues of a static $SU(2)$ Polyakov line at high temperatures. In the $T \to \infty$ limit dimensional reduction takes place and perturbation theory works well. The Polyakov loop has values at the center of the gauge group. If one lowers the temperature, however, the fluctuations of the Polyakov loop around these perturbative values increase. We studied the fluctuations of a static P in the whole range of its possible variation and found the 1-loop effective action for its eigenvalues, interacting in a covariant way with the A_i fields. We found that while the kinetic energy in the magnetic sector is center-symmetric, the kinetic energy in the electric sector is not. If one wishes to preserve this symmetry one would have to sum over all powers of the more general time-dependent electric field. For small values of A_4 all functions are singular and behave as $1/A_4$, which is due to the contribution of the zero Matsubara frequency.

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